PRACTICE MIDTERM 2 − SOLUTIONS

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1. (a) State carefully: The Mean Value Theorem

Mean Value Theorem: Let f be a function that satisfies the following two hypotheses:

1) f is continuous on the closed interval $[a, b]$
2) f is differentiable on the open interval (a, b)

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

(b) A particle moves in a straight line with acceleration at time t given by $a(t) = 2t\ m/\sec^2$. It has initial velocity $v(0) = 5\ m/\sec$. What is the net change in position between $t = 0$ and $t = 3$?

Our goal is to calculate $s(3) - s(0)$.

By antidifferentiating $a(t) = 2t$, we get $v(t) = t^2 + C$. But $v(0) = 5$, so $0^2 + C = 5$, so $C = 5$. Hence, $v(t) = t^2 + 5$.

Now by antidifferentiating $v(t) = t^2 + 5$, we get $s(t) = \frac{t^3}{3} + 5t + C'$

Hence $s(3) - s(0) = \frac{3^3}{3} + 15 + C' - (0 + 0 + C') = 24 + C' - C' = 24$.

So $\boxed{s(3) - s(0) = 24 \text{ m}}$

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2. Find the equation of the line tangent to the curve $2(x^2+y^2)^2 = 25(x^2-y^2)$ at the point $(3,1)$

   Using implicit differentiation, we get:

$$4(x^2+y^2)(2x+2yy') = 25(2x-2yy')$$

   And plugging in $x=3$ and $y=1$, we get:

$$4(9+1)(6+2y') = 25(6-2y')$$

   That is:

$$240+80y' = 150-50y'$$

   So solving for y', we get:

$$y' = -\frac{9}{13}$$

   Hence, an equation of the tangent line to the curve at $(3,1)$ is: $\boxed{y-1 = -\frac{9}{13}(x-3)}$

3. (a) Let $h(x) = f(g(x))$, where f, f', g, g' are differentiable everywhere. Find $h''(0)$, given that $f'(2) = 5$, $f''(2) = 4$, $g(0) = 2$, $g'(0) = 3$, and $g''(0) = 2$.

We have:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

Hence:

$$\begin{aligned} h''(x) &= \left(f'(g(x))g'(x) \right)' \\ &= \left(f'(g(x)) \right)' g'(x) + f'(g(x)) \cdot g''(x) \quad \text{(By the product rule)} \\ &= f''(g(x))g'(x) \cdot g'(x) + f'(g(x))g''(x) \quad \text{(By the chain rule)} \\ &= f''(g(x)) \cdot \left(g'(x)\right)^2 + f'(g(x)) \cdot g''(x) \end{aligned}$$

In particular:

$$\begin{aligned} h''(0) &= f''(g(0)) \cdot \left(g'(0)\right)^2 + f'(g(0)) \cdot g''(0) \\ &= f''(2) \cdot (3)^2 + f'(2) \cdot 2 \\ &= 4 \times 9 + 5 \times 2 \\ &= 46 \end{aligned}$$

Therefore $\boxed{h''(0) = 46}$

(b) Use differentials to estimate the amount of paint needed to apply at coat of paint 0.05 cm thick to a hemispherical dome with diameter 50 cm.

We know that $V = \frac{2}{3}\pi r^3$ (remember that we have a **hemi**sphere here!), so $dV = \frac{2}{3} 3\pi r^2 dr = 2\pi r^2 dr$.

Now the amount of paint needed is $V(r + dr) - V(r) = \Delta V \approx dV = 2\pi r^2 dr$.

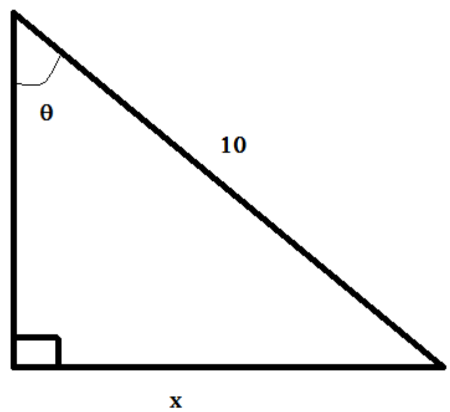
Here $dr = 0.05$, and $r = 25$ (remember that the **diameter**, not the radius, is 50 cm thick!), hence:

$$\boxed{dV = 2\pi(25)^2(0.05) = \frac{125\pi}{2}\ cm^3}$$

4. A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of $2ft/s$, how fast is the angle between the top of the ladder and the wall changing when that angle is $\frac{\pi}{4}$ radians?

   1) Picture:

   1A/Practice Exams/Triangle.png



   2) We want to find $\frac{d\theta}{dt}$
   3) We have $\sin(\theta) = \frac{x}{10}$
   4) Differentiating with respect to t, we get: $\cos(\theta)\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$
   5) We know $\theta = \frac{\pi}{4}$ and $\frac{dx}{dt} = 2$, so $\cos(\frac{\pi}{4})\frac{d\theta}{dt} = \frac{2}{10}$, so $\frac{\sqrt{2}}{2}\frac{d\theta}{dt} = \frac{1}{5}$, so $\boxed{\frac{d\theta}{dt} = \frac{2}{5\sqrt{2}} = \frac{\sqrt{2}}{5} \text{ rad/s}}$

5. Compute

(a)

$$\lim_{x\to\infty}(xe^{\frac{1}{x}}-x) = \lim_{x\to\infty} x(e^{\frac{1}{x}}-1) = \lim_{x\to\infty}\frac{e^{\frac{1}{x}}-1}{\frac{1}{x}} \overset{H}{=} \lim_{x\to\infty}\frac{e^{\frac{1}{x}}(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\infty} e^{\frac{1}{x}} = e^0 = 1$$

where $\overset{H}{=}$ means 'by l'Hopital's rule' (which is legitimate, because we have the indeterminate form $\frac{0}{0}$)

(b) $\frac{d}{dt}\left(t^{\sin(t)}\right)$

This is just logarithmic differentiation:

1) Let $y = t^{\sin(t)}$
2) $\ln(y) = \sin(t)\ln(t)$
3) $\frac{y'}{y} = \cos(t)\ln(t) + \frac{\sin(t)}{t}$
4) $y' = \left(\cos(t)\ln(t) + \frac{\sin(t)}{t}\right) y = \left(\cos(t)\ln(t) + \frac{\sin(t)}{t}\right) t^{\sin(t)}$

So $\boxed{\frac{d}{dt}\left(t^{\sin(t)}\right) = \left(\cos(t)\ln(t) + \frac{\sin(t)}{t}\right) t^{\sin(t)}}$

(c)

$$\lim_{t\to 0^+}\frac{2\sqrt{t}}{\arcsin(t)} \overset{H}{=} \lim_{t\to 0^+}\frac{\frac{2}{2\sqrt{t}}}{\frac{1}{\sqrt{1-t^2}}} = \lim_{t\to 0^+}\frac{\sqrt{1-t^2}}{\sqrt{t}} = \frac{1}{0^+} = \infty$$

Also, the use of l'Hopital's rule is legitimate because we have the indeterminate form $\frac{0}{0}$

6. Let $h(x) = 3x^5 - 5x^3 + 3$

(a) Find the intervals of increase and decrease

$h'(x) = 15x^4 - 15x^2 = 15x^2(x^2 - 1) = 15x^2(x-1)(x+1)$

Then $h'(x) = 0$ if and only if $x = 0$ or $x = \pm 1$.

Now, by using the following sign table (this is just one way of doing this problem, if you find the right answer using another way, that's fine too!), we get:

- h is increasing on $(-\infty, -1) \cup (1, \infty)$
- h is decreasing on $(-1, 0) \cup (0, 1)$ ($(-1, 1)$ is also a valid answer)

1A/Practice Exams/Steeltable1.png

x	-oo		-1		0		1		oo
x²		+		+	0	+		+	
x - 1		−		−		−	0	+	
x + 1		−	0	+		+		+	
h'(x)		+	0	−	0	−	0	+	
h(x)		↗	5	↘	3	↘	1	↗	

(b) Find the local maxima and minima

The only critical numbers of h are 0 and ± 1. By the first derivative test, $\boxed{h(-1) = -3 + 5 + 3 = 5}$ is a local maximum (h' changes sign from positive to negative there), and $\boxed{h(1) = 3 - 5 + 3 = 1}$ is a local minimum (h' changes sign from negative to positive).

Note: Notice that $h(0) = 3$ is neither a local maximum or minimum, since h' does not change sign at 0

(c) Find the intervals of concavity and inflection points

$$h''(x) = 60x^3 - 30x = 60x(x^2 - \tfrac{1}{2}) = 60x(x - \sqrt{\tfrac{1}{2}})(x + \sqrt{\tfrac{1}{2}}).$$

Then $h''(x) = 0$ if and only if $x = 0$ or $x = \pm\sqrt{\frac{1}{2}}$

Again, using a sign table, we can conclude the following:

- h is concave up on $(-\sqrt{\frac{1}{2}}, 0) \cup (\sqrt{\frac{1}{2}}, \infty)$
- h is concave down on $(-\infty, -\sqrt{\frac{1}{2}}) \cup (0, \sqrt{\frac{1}{2}})$.

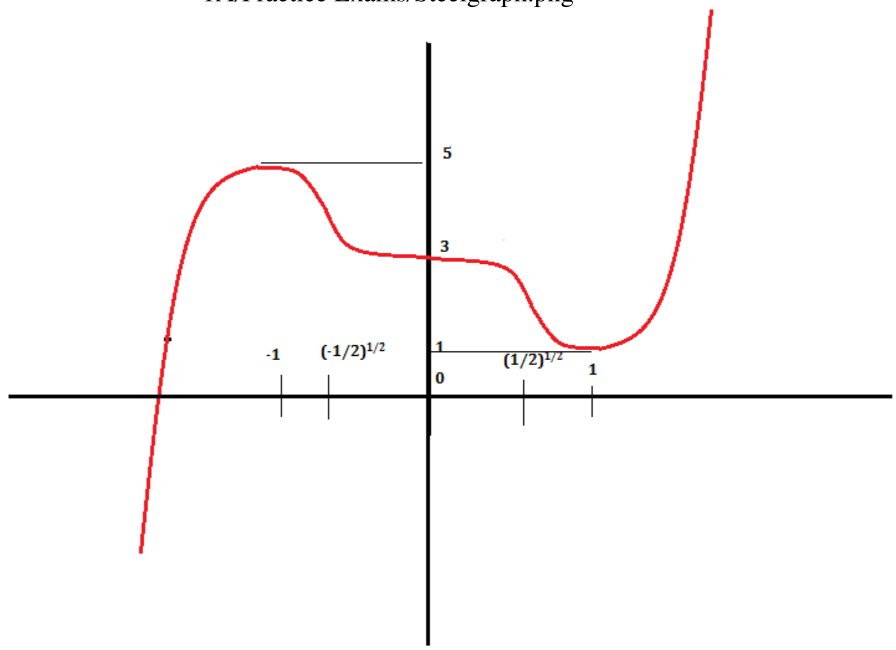
In particular, $(-\sqrt{\frac{1}{2}}, h(-\sqrt{\frac{1}{2}})) = (-\sqrt{\frac{1}{2}}, -\frac{3}{4\sqrt{2}} + \frac{5}{2\sqrt{2}} + 3)$, $(0, h(0)) = (0, 3)$, and $(\sqrt{\frac{1}{2}}, h(\sqrt{\frac{1}{2}})) = (\sqrt{\frac{1}{2}}, \frac{3}{4\sqrt{2}} - \frac{5}{2\sqrt{2}} + 3)$ are inflection points of h.

1A/Practice Exams/Steeltable2.png

x	-oo		$(-1/2)^{1/2}$		0		$(1/2)^{1/2}$		oo
x		–		–	0	+		+	
x - $(1/2)^{1/2}$		–		–		–	0	+	
x + $(1/2)^{1/2}$		–	0	+		+		+	
h''(x)		–	0	+	0	–	0	+	
h(x)		C.D	I.P	C.U	I.P	C.D	I.P	C.U	

(d) Sketch the graph of h

1A/Practice Exams/Steelgraph.png



Note: The graph is not drawn to scale for more legibility.

7. A cylindrical can without a top is made to contain $V\ cm^3$ of liquid. Find the dimensions that will minimize the amount of metal required to make the can.

   1) The surface area of the can is $S = \pi r^2 + 2\pi r h$, where r is the radius of the can, and h is the height of the can (basically, S = area of the bottom disk + the area of the (cylindrical) side)

   2) Now, we know that the volume of the can, $\pi r^2 h$ is constant, equal to V. So $\pi r^2 h = V$, so $h = \frac{V}{\pi r^2}$

      In particular, we may rewrite the surface area as a function of r only, $S(r) = \pi r^2 + 2\pi r(\frac{V}{\pi r^2})$, so $\boxed{S(r) = \pi r^2 + 2\frac{V}{r}}$

   3) The only constraint is that $r > 0$

   4) $S'(r) = 2\pi r - 2\frac{V}{r^2} = 0 \Leftrightarrow 2\pi r = 2\frac{V}{r^2} \Leftrightarrow 2\pi r^3 = 2V$ (cross-multiplying) $\Leftrightarrow \boxed{r = \sqrt[3]{\frac{V}{\pi}}}$.

      Now $S'(r) < 0$ if $r < \sqrt[3]{\frac{V}{\pi}}$ and $S'(r) > 0$ if $r > \sqrt[3]{\frac{V}{\pi}}$, so by the first derivative test for absolute extreme values, $r = \sqrt[3]{\frac{V}{\pi}}$ is the minimizer of V.

   5) In particular, the dimensions that will minimize the amount of metal required to make the can are: $\boxed{r = \sqrt[3]{\frac{V}{\pi}}}$ and $\boxed{h = \frac{V}{\pi r^2} = \frac{V}{\pi \frac{V^{\frac{2}{3}}}{\pi^{\frac{2}{3}}}} = \sqrt[3]{\frac{V}{\pi}}}$

8. A sample of tritium-3 decayed to 94.5 percent of its original amount after one year.

   (a) What is the half-life of tritium-3?

   Let $m(t)$ be the mass of tritium-3 that remains after t years. Since this problem deals with radioactive decay, we know that $m(t)$ satisfies the differential equation $m'(t) = km(t)$ for some k, whose solution is:

$$m(t) = Ce^{kt}$$

   where C and k are constants.

   We know that after one year, the sample decayed to 94.5 percent of its original amount, therefore $m(1) = 0.945m(0)$, that is:

$$\begin{aligned} Ce^{k(1)} &= 0.945Ce^{k(0)} \\ e^k &= 0.945 \\ k &= \ln(0.945) \end{aligned}$$

   Therefore $m(t) = Ce^{\ln(0.945)t} = C(0.945)^t$.

   To find the half-life of tritium-3, we are looking for the time t^* where $m\left(t^*\right) = \frac{m(0)}{2}$, that is:

$$\begin{aligned} C(0.945)^{t^*} &= \frac{C(0.945)^0}{2} \\ C(0.945)^{t^*} &= \frac{C}{2} \\ (0.945)^{t^*} &= \frac{1}{2} \\ t^* \ln(0.945) &= \ln\left(\frac{1}{2}\right) = -\ln(2) \\ t^* &= \frac{-\ln(2)}{\ln(0.945)} \approx 12.25 \end{aligned}$$

   So the half-life of tritium-3 is $\boxed{\frac{-\ln(2)}{\ln(0.945)} \approx 12.25}$ years.

(b) How long would it take for the sample to decay to 20 percent of its original amount?

Now we are looking for the time t^* where $m\left(t^*\right) = \frac{m(0)}{5}$, that is:

$$\begin{aligned} C(0.945)^{t^*} &= \frac{C(0.945)^0}{5} \\ C(0.945)^{t^*} &= \frac{C}{5} \\ (0.945)^{t^*} &= \frac{1}{5} \\ t^* \ln(0.945) &= \ln\left(\frac{1}{5}\right) = -\ln(5) \\ t^* &= \frac{-\ln(5)}{\ln(0.945)} \approx 28.45 \end{aligned}$$

So it would take about $\boxed{\frac{-\ln(5)}{\ln(0.945)} \approx 28.45}$ years for tritium-3 to decay to 20 percent of its original amount.